

# Ignition of superconducting vortices by acoustic standing waves

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February 1, 2008*

Nucleation of vortices in a superconductor below the first critical field can be assisted by transverse sound in the GHz frequency range. Vortices will enter and exist the superconductor at the frequency of the sound. We compute the threshold parameters of the sound and show that this effect is within experimental reach.

PACS numbers: 74.25.Ld, 74.25.Qt, 74.90.+n

A superconducting cylinder rotated at an angular velocity  $\Omega$  about its symmetry axis develops a magnetic moment  $\mathbf{M} = -(mc/2\pi e)\Omega$ , where  $m$  and  $e$  are bare electron mass and charge, and  $c$  is the speed of light. This effect predicted by London [1] has been subsequently tested in experiment and proved with an accuracy to many significant figures. It is a consequence of a more general gyromagnetic effect predicted by Barnett [2]: “A substance which is magnetic must become magnetized by a sort of molecular gyroscopic motion on receiving an angular velocity”. Barnett effect is, in its turn, a consequence of the Larmor theorem [3]: In the rotating frame of reference the action of the rotation on charged particles is equivalent to the action of the magnetic field  $\mathbf{H}_\Omega = \Omega/\gamma$  [3], where  $\gamma$  is the gyromagnetic ratio. For electron’s orbital motion  $\gamma = e/(2mc) \approx 0.9 \times 10^7 (\text{Gauss})^{-1} \text{ s}^{-1}$ . Thus, in practice, the fictitious field in the reference frame of a rotating macroscopic cylinder can hardly exceed a fraction of a milligauss. This would be well below the lower critical field  $\mathbf{H}_{c1}$  when the temperature of the superconductor is not too close to  $T_c$ . Due to the Meissner effect [4] (considered in the frame of the rotating cylinder) such a field would be expelled from the bulk of the cylinder by a superconducting current induced at the surface. Writing  $\mathbf{B} = \mathbf{H}_\Omega + 4\pi\mathbf{M} = 0$  for the total field in the bulk, one obtains the London’s magnetic moment,  $\mathbf{M} = -\mathbf{H}_\Omega/(4\pi) = -(mc/2\pi e)\Omega$ . Due to the symmetry of the problem it is the same in the rotating and laboratory frames.

In this Letter we would like to take this problem a little further and look at the consequence of an angular velocity well beyond the experimental limit. In particular, we are interested in the rotational velocity of a magnitude that would generate a fictitious magnetic field that exceeds  $\mathbf{H}_{c1}$ . If Larmor’s theorem still holds, than it must be the case that a superconducting vortex enters the bulk of the cylinder. This would require the angular velocity to be of order  $10^9 \text{ s}^{-1}$ , clearly surpassing the feasible experimental value for a mechanical rotation. While this scenario is merely a thought experiment we will use it as a motivation to study the effect of local rotations generated in a superconductor by high frequency ultrasound.

Interaction of sound with vortices has been studied in the past [5, 6, 7]. Radiation of phonons by supersonic vortices [8, 9], phonon contribution to the vortex mass [10, 11, 12], and decoherence of flux qubits by phonons [13, 14] have been investigated. In this Letter we are addressing a completely different problem – possibility of the nucleation of a vortex by sound.

Within continuous elastic theory, local deformations are described by the displacement vector field,  $\mathbf{u}(\mathbf{r}, t)$ . We will be interested in the effect of transverse sound waves. Such waves create shear deformations of the crystal lattice, such that

$$\nabla \cdot \mathbf{u} = 0. \quad (1)$$

In the long-wave limit they do not affect the density of the ionic lattice but result in a local rotation at an angular velocity [15]

$$\Omega(\mathbf{r}, t) = \frac{1}{2} \nabla \times \dot{\mathbf{u}}. \quad (2)$$

The frequency of ultrasound achievable in experiment with, e.g., surface acoustic waves can easily be in the ballpark of  $f \sim 10^{10} \text{ s}^{-1}$  [16]. According to Eq. (2) a sound of such frequency and amplitude of a few nanometers can provide  $\Omega \sim 10^9 \text{ s}^{-1}$  that can generate fictitious magnetic fields above  $\mathbf{H}_{c1}$ . For practical purposes, it may be convenient to loosen the restriction on the frequency and amplitude of ultrasound by applying an external magnetic field  $\mathbf{H}$  near, but less than,  $\mathbf{H}_{c1}$ . We shall see that within one percent of  $\mathbf{H}_{c1}$ , vortices can be ignited by the ultrasound in the GHz frequency range.

For a vortex to enter a superconductor, the Gibbs free energy of the system must be lowered. We compute the extra free energy due to the vortex and determine the condition at which it becomes negative. It should be noted that the system under consideration is dynamical, and therefore is not at a thermodynamical equilibrium. However, we are interested in the free energy of the Cooper pairs which can adjust to the changes of state in a time scale orders of magnitude shorter than the period of the sound. This time scale is proportional to the

relaxation time  $\tau$  of the cooper pairs, i.e.  $\tau \sim 10^{-12}$ s. As mentioned before, the period of the sound  $T = 1/f$  will be always greater than  $10^{-10}$ s. Under these conditions, our system is adiabatic and the thermodynamic equilibrium can be safely established. The calculation that follows is similar to the conventional calculation of  $\mathbf{H}_{c1}$ . The presence of  $\Omega$ , however, introduces a new feature into this calculation so we will follow it all the way through to show how the sound enters the problem.

It is convenient to calculate the extra free energy in terms of the magnetic field and its spacial derivatives. The electric field produced by the time derivatives will be neglected. The kinetic energy of the superfluid is  $\frac{1}{2}n_s m \mathbf{v}^2$  where  $n_s$  is the number density of the superconducting electrons and

$$\mathbf{v} = \frac{e^*}{m^*c} \left( \frac{\hbar}{e^*} \nabla \varphi - \mathbf{A} \right) \quad (3)$$

is the velocity of the cooper pairs with  $\varphi$  and  $\mathbf{A}$  being the phase of the superfluid wavefunction and the magnetic vector potential, respectively. The stated quantities represent the effective mass and charge of cooper pairs. We will take them to be  $m^* = 2m$  and  $e^* = 2e$ . The normal electrons experience viscous forces as they move relative to the nuclei contributing zero average normal current. The ionic charge per unit volume consisting of the nuclei and the normal electrons is therefore exactly opposite to that of the cooper pairs. The total current is then

$$\mathbf{j} = en_s(\mathbf{v} - \dot{\mathbf{u}}), \quad (4)$$

where  $\dot{\mathbf{u}}$  is the velocity of ions. Eq. (4) reflects the fact that the electric current corresponds to the motion of electrons relative to ions. It is invariant with respect to the motion of the reference frame. With Eq. (3) in mind we can write the gauge invariant current in terms of  $\varphi$  and  $\mathbf{u}$  as

$$\mathbf{j} = \frac{n_s e \hbar}{2m} \left( \nabla \varphi - \frac{2e}{\hbar} \mathbf{A}_{eff} \right), \quad (5)$$

where

$$\mathbf{A}_{eff} = \mathbf{A} + \frac{mc}{e} \dot{\mathbf{u}} \quad (6)$$

is the effective vector potential felt by the electrons in the rotating frame of the ions [12]. In terms of the total current  $\mathbf{j}$ , the kinetic energy of the superconducting electrons may be expressed in the form

$$KE_e = \int d^3r \frac{n_s m}{2} \left( \frac{1}{n_s e} \mathbf{j} + \dot{\mathbf{u}} \right)^2. \quad (7)$$

The energy of the sound is

$$E_s = \int d^3r \frac{1}{2} (\rho_0 \dot{\mathbf{u}}^2 - \lambda_{iklm} u_{ik} u_{lm}) \quad (8)$$

in which  $\rho_0$  is the combined mass density of ions and normal electrons,  $\lambda_{iklm}$  is the tensor of elastic coefficients and  $u_{ik} = \frac{1}{2}(\partial_i u_k + \partial_k u_i)$  is the strain tensor. Using Maxwell's equation  $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{j}$  and combining Eqs. (7) and (8) the expression for the total Gibbs free energy yields

$$\begin{aligned} G = & \mathcal{F}_0 + \frac{1}{8\pi} \int d^3r \left[ \mathbf{B}^2 + \frac{\lambda^2}{f(r)} (\nabla \times \mathbf{B})^2 \right] \\ & + \frac{1}{4\pi} \int d^3r \frac{mc}{e} \dot{\mathbf{u}} \cdot (\nabla \times \mathbf{B}) - \frac{1}{4\pi} \int d^3r \mathbf{H} \cdot \mathbf{B} \\ & + \int d^3r \frac{1}{2} [\rho_0 \dot{\mathbf{u}}^2 - \lambda_{iklm} u_{ik} u_{lm}], \end{aligned} \quad (9)$$

Here,  $\mathcal{F}_0$  is the free energy in the absence of currents, fields, and sound,  $\lambda = \sqrt{mc^2/4\pi n_s e^2}$  is the London penetration depth,  $f(r) = (|\psi|/|\psi_\infty|)^2$  in which  $\psi$  is the complex order parameter and  $|\psi_\infty| = \sqrt{n_s/2}$  is the order parameter in the absence of gradients and fields, and  $\rho = \rho_0 + n_s m$  is the total mass density of the superconductor. The fourth term can be recognized as the interaction of the external magnetic field with the magnetization. It is this term that is responsible for the nucleation of vortices in the absence of sound when  $H \geq H_{c1}$ .

Before we can calculate the free energy of Eq. (9) we must first work out the magnetic field. This can be done by replacing the current in the Maxwell's equation  $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{j}$  with Eq. (5) and defining a gauge invariant vector potential  $\mathbf{Q} = \mathbf{A} - (\hbar c/2e)\nabla \varphi$ , so that we obtain the following equation:

$$\lambda^2 \nabla \times (\nabla \times \mathbf{Q}) + f(r) \mathbf{Q} = -\frac{mc}{e} f(r) \dot{\mathbf{u}}. \quad (10)$$

For  $\nabla \varphi = 0$  ( $\mathbf{Q} = \mathbf{A}$ ) Eq. (10) becomes equivalent to the London's equation with a source. When a vortex enters a superconductor the phase must be quantized according to the condition  $\oint \nabla \varphi \cdot d\mathbf{l} = 2\pi$ . For certainty we consider a transverse standing sound wave having one node at the center of a superconducting slab of thickness  $d$  large compared to the coherence length  $\xi$ . The external field is applied parallel to the slab, see Fig. 1. In this case  $\lambda_s = 2d$ . Generalization to standing waves with many nodes is straightforward. If the field is close to  $\mathbf{H}_{c1}$ , a vortex will periodically enter and exit the slab. The boundary condition on the current is  $\mathbf{J}_\perp \cdot \mathbf{n} = 0$ , where  $\mathbf{n}$  is the direction of the surface. If the thickness of the slab  $d$  is of order or less than  $\lambda$ , this boundary will distort the cylindrical symmetry of the vortex. We can satisfy the boundary condition by placing image vortices of alternating sign a distance  $d$  apart on the outside of the slab. The equation for the magnetic field, in the region  $r > \xi$  where  $|\psi| = 1$ , can then be written in two parts, namely  $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_v$ , such that the first term satisfies

$$\lambda^2 \nabla \times (\nabla \times \mathbf{B}_0) + \mathbf{B}_0 = -\frac{2mc}{e} \Omega, \quad (11)$$

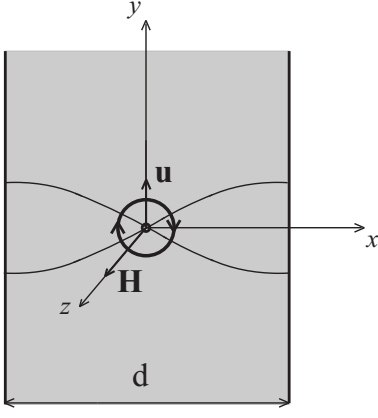


FIG. 1: Standing wave in a slab with one node at the center. The vortex is generated at the node where  $\Omega$  is maximum.

while the second is a solution of

$$\lambda^2 \nabla \times (\nabla \times \mathbf{B}_v) + \mathbf{B}_v = \Phi_0 \mathbf{e}_z \sum_{n=-\infty}^{\infty} (-1)^n \delta(\mathbf{r} + n d \mathbf{e}_x), \quad (12)$$

where  $\Phi_0 = hc/2e$  is the flux quantum. Notice that Eqs. (11) and (12) can be obtained by taking a curl of Eq. (10) with the account of the vortex cores represented by the delta functions.

Since we are interested in standing sound waves we can choose the displacement vector  $\mathbf{u}$  to be

$$\mathbf{u}(\mathbf{r}, t) = u_0 \sin(kx) \sin(\omega t) \mathbf{e}_y. \quad (13)$$

The quantity  $k = \omega/v = 2\pi/\lambda_s = \pi/d$  is the wave number with  $\lambda_s$  and  $v$  being the wavelength and the speed of sound respectively. It is easy to see from Eq. (2) that  $\Omega$  is maximum at the nodes. The corresponding solutions to Eqs. (11) and (12) with the boundary condition  $\mathbf{B} = \mathbf{H}$  at  $x = \pm d/2$  are

$$\mathbf{B}_0(x) = \mathbf{B}_M + \mathbf{B}_s \quad (14)$$

$$\mathbf{B}_v(\mathbf{r}) = \sum_{n=-\infty}^{\infty} (-1)^n \mathbf{b}(\mathbf{r}_n) \quad (15)$$

where

$$\mathbf{B}_M = 2\mathbf{H} \frac{\sinh(d/2\lambda)}{\sinh(d/\lambda)} \cosh\left(\frac{x}{\lambda}\right) \quad (16)$$

$$\mathbf{B}_s = -\frac{2mc}{e} \frac{\Omega}{1 + k^2 \lambda^2} \quad (17)$$

$$\mathbf{b}(\mathbf{r}_n) = \frac{\Phi_0}{2\pi\lambda^2} K_0(|\mathbf{r} + n d \mathbf{e}_x|/\lambda) \mathbf{e}_z, \quad (18)$$

and  $K_0$  is a zeroth-order Hankel function of imaginary argument. The first term in Eq. (14) is the Meissner field while the second is due to the sound.

Let us now integrate by parts the third term in Eq. (9) and insert  $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_v$ . By doing so we obtain

$$G = G_0 + \Delta E + \frac{1}{4\pi} \int d^3r \mathbf{B}_v \cdot \left[ \frac{2mc}{e} \Omega + \mathbf{B}_s \right]$$

$$+ \frac{1}{4\pi} \int d^3r \frac{\lambda^2}{f(r)} (\nabla \times \mathbf{B}_v) \cdot (\nabla \times \mathbf{B}_s) \quad (19)$$

where  $G_0$  is the Gibbs free energy without a vortex and

$$\begin{aligned} \Delta E = & \frac{1}{4\pi} \int d^3r \frac{\lambda^2}{f(r)} (\nabla \times \mathbf{B}_M) \cdot (\nabla \times \mathbf{B}_v) \\ & + \frac{1}{8\pi} \int d^3r \left[ \mathbf{B}_v^2 + \frac{\lambda^2}{f(r)} (\nabla \times \mathbf{B}_v)^2 \right] \\ & - \frac{1}{4\pi} \int d^3r \mathbf{H} \cdot [\mathbf{B}_v + \mathbf{B}_M] \end{aligned} \quad (20)$$

is the vortex energy. One can simplify the volume integrals in Eq. (19) by separating the integration over the core from the integration over the volume outside of the core. When the latter is integrated by parts, the integrals outside the core cancel, and the free energy in Eq. (19) with the help of Eq. (11) becomes

$$\Delta \mathcal{F} = \Delta \mathcal{F}_1 + \Delta \mathcal{F}_2 + \Delta \mathcal{F}_3 + \Delta E, \quad (21)$$

where

$$\Delta \mathcal{F}_1 = \frac{1}{4\pi} \int_c d^3r \mathbf{B}_v \cdot \left[ \frac{2mc}{e} \Omega + \mathbf{B}_s \right] \quad (22)$$

$$\Delta \mathcal{F}_2 = \frac{\lambda^2}{4\pi} \oint_c \mathbf{B}_v \times (\nabla \times \mathbf{B}_s) \cdot d\mathbf{s} \quad (23)$$

$$\Delta \mathcal{F}_3 = \frac{1}{4\pi} \int_c d^3r \frac{\lambda^2}{f(r)} (\nabla \times \mathbf{B}_v) \cdot (\nabla \times \mathbf{B}_s). \quad (24)$$

The subscript  $c$  indicates an integration over the core. The surface integral in Eq. (23) is over the boundary of the normal core. Near the vortex core  $f(r) = (r/a)^2$ , where  $a \approx \xi$ . It is straightforward to check that in the limit  $r \rightarrow 0$  the exact solution to Eq. (10) for the vector potential  $\mathbf{A}_s(\mathbf{r})$  is

$$\mathbf{A}_s(\mathbf{r}) = -\frac{2mc}{e} \Omega_0 x \mathbf{e}_y. \quad (25)$$

Then the magnetic field  $\mathbf{B}_s = \nabla \times \mathbf{A}_s(\mathbf{r})$  generated by the sound at the center of the core is

$$\mathbf{B}_s(\mathbf{r}) = -\frac{2mc}{e} \Omega_0 \mathbf{e}_z, \quad (26)$$

where

$$\Omega_0 = \frac{1}{2} u_0 k \omega = \frac{\pi}{2} \frac{u_0}{d} \omega. \quad (27)$$

It can be shown that near the vortex core,  $\nabla \times \mathbf{B}_v \propto r^4$  and  $\nabla \times \mathbf{B}_s \propto r^5$ . The expression under the integral in Eq. (24) is therefore proportional to  $r^8$  near the center of the core and to  $r K_1(r/\lambda)$  at  $r \gtrsim \xi$ . Thus, the integral in Eq. (24) falls off very rapidly inside the core and can be neglected.

The case of  $k\lambda \geq 1$  is rather involved as it requires explicit knowledge of the structure of the vortex core.

For  $k\lambda \ll 1$  Eq. (17) provides that  $\mathbf{B}_s \cong -(2mc/e)\mathbf{\Omega}$  in all regions of space, so that  $\Delta\mathcal{F}_1 \rightarrow 0$ . In this limit the Meissner field  $\mathbf{B}_M$  and the fields due to images can be neglected. The total interaction energy per unit length of the vortex acquires the simplest form at  $\kappa = \lambda/\xi \gg 1$ :

$$\frac{\Delta\mathcal{F}_2}{L} = -\frac{mc}{2\pi}\Omega_0\Phi_0\left(\frac{k\lambda}{\kappa}\right)^2 \ln \kappa, \quad (28)$$

where  $L$  is the dimension of the slab in the  $z$ -direction.

If one excludes small contribution from the vortex core in Eq. (20), then the integration by parts yields

$$\frac{\Delta E}{L} = \frac{\lambda^2}{8\pi} \oint \mathbf{B}_v \times (\nabla \times \mathbf{B}_v) \cdot d\mathbf{s} - \frac{1}{4\pi} \int d^3r \mathbf{H} \cdot \mathbf{B}_v. \quad (29)$$

This approximation is good if  $\lambda$  and  $d$  are large compared to the coherence length  $\xi$ . Then the vortex energy per unit length is

$$\frac{\Delta E}{L} = \frac{\Phi_0^2}{(4\pi\lambda)^2} \ln \kappa - \frac{\Phi_0 H}{4\pi}. \quad (30)$$

The first term in this expression is the self-energy of the vortex, while the second term is the energy of the interaction of the flux quantum with the external field.

The condition for the nucleation of the vortex,  $\Delta\mathcal{F}_2 + \Delta E = 0$ , yields

$$\frac{2mc}{e}\Omega_0\left(\frac{k\lambda}{\kappa}\right)^2 \ln \kappa = \epsilon H_{c1}, \quad (31)$$

where

$$\epsilon = 1 - \frac{H}{H_{c1}} \quad (32)$$

and  $H_{c1} = \Phi_0 \ln \kappa / (4\pi\lambda^2)$  is the first critical field that follows from Eq. (30) at  $\Delta E = 0$ . Substituting Eq. (27) into Eq. (31), one finds the conditions on the frequency  $f$  and amplitude  $u_0$  of the sound needed to nucleate a vortex in the geometry shown in Fig. 1:

$$f = \frac{v}{2d}, \quad u_0 = \frac{\epsilon}{4} \left( \frac{d}{\pi\lambda} \right)^4 \frac{\hbar\kappa^2}{mv}. \quad (33)$$

While the last formula was derived under the conditions  $\pi\xi < \pi\lambda \ll d$ , our numerical analysis shows that it holds even for  $d \sim \pi\lambda$  at  $\kappa \gg 1$  and is true by order of magnitude for  $\kappa \sim 1$ . In this case the expression for  $H_{c1}$  carries the signature of the surface barrier [17]:  $H_{c1} = \beta\Phi_0 \ln \kappa / (4\pi\lambda^2)$ , where

$$\beta = \frac{1 - 2(\ln \kappa)^{-1} \sum_{n=1}^{\infty} (-1)^n K_0(dn/\lambda)}{1 - 2\sinh(d/2\lambda)[\sinh(d/\lambda)]^{-1}}. \quad (34)$$

For the speed of the transverse sound  $v \sim 3 \times 10^5 \text{cm/s}$ , in a slab of thickness  $d \sim \pi\lambda \sim 6 \times 10^{-5} \text{cm}$  and  $\kappa \sim 2$ ,

with  $H$  within one percent of  $H_{c1}$ , one gets from Eq. (33)  $f \sim 3 \text{GHz}$  and  $u_0 \sim 0.2 \text{nm}$ . These are accessible values of frequency and amplitude of ultrasound.

As  $\mathbf{\Omega}$  changes its sign every half a period of the sound, vortices are periodically attracted and repelled by the standing acoustic wave in Fig. 1. Periodic entering and expulsion of vortices should result in the elevated attenuation of the ultrasound and in the ac voltage across the slab at the sound frequency. In a different experiment one can assist vortices to enter or exit the superconductor with the help of the surface acoustic waves (SAW). Like in the problem with a slab, local rotation of the crystal produced by the SAW may assist nucleation of the vortex at the field just below  $H_{c1}$ .

In conclusion, we have demonstrated that nucleation of a vortex in a superconductor can be assisted by ultrasound. In the presence of a standing sound wave, vortices will periodically enter and exit the superconductor. The required amplitude and frequency of ultrasound are within experimental reach.

We thank Lev Bulaevskii and Carlos Calero for useful discussions. This work has been supported by the Department of Energy through Grant No. DE-FG02-93ER45487.

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- [1] F. London, *Superfluids* (Wiley, New York, 1950), Vol. 1.
  - [2] S. J. Barnett, Phys. Rev. **6**, 239 (1915).
  - [3] J. Larmor, Lond. Math. Soc. Proc. **1**, 1 (1903).
  - [4] W. Meissner and R. Ochsenfeld, Naturwiss. **21**, 787 (1933).
  - [5] H. Haneda and T. Ishiguro, Physica **C 235-240**, 2076 (1994).
  - [6] D. Dominguez, L. Bulaevskii, B. Ivlev, M. Maley, and A. R. Bishop, Phys. Rev. Lett. **74**, 2579 (1995); Phys. Rev. B **51**, 15649 (1995); Phys. Rev. B **53**, 6682 (1996).
  - [7] E. B. Sonin, Phys. Rev. Lett. **76**, 2794 (1996).
  - [8] B. I. Ivlev, S. Mejia-Rosales, and M. N. Kunchur, Phys. Rev. B **60**, 12419 (1999).
  - [9] L. N. Bulaevskii and E. M. Chudnovsky, Phys. Rev. B **72**, 094518 (2005).
  - [10] M. W. Coffey, Phys. Rev. B **49**, 9774 (1994).
  - [11] J.-M. Duan and E. Simanek, Phys. Lett. **A 190**, 118 (1994).
  - [12] E. M. Chudnovsky and A. B. Kuklov, Phys. Rev. Lett. **91**, 067004 (2003).
  - [13] E. M. Chudnovsky and A. B. Kuklov, Phys. Rev. B **67**, 064515 (2003).
  - [14] J. Albert and E. M. Chudnovsky, Phys. Rev. B **75**, 144502 (2007).
  - [15] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, New York, 1959).
  - [16] M. M. de Lima, Jr. and P. V. Santos, Rep. Prog. Phys. **68**, 1639 (2005).
  - [17] C. P. Bean, and J. D. Livingston, Phys. Rev. Lett. **12**, 1 (1963).